Two-Dimensional Equations for Electroelastic Plates with Relatively Large Shear Deformations

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***Abstract*—A set of two-dimensional, nonlinear equations for electroelastic plates in moderately large thickness-shear deformations is obtained from the variational formulation of the three-dimensional equations of nonlinear electroelasticity by expanding the mechanical displacement vector and the electric potential into power series in the plate thickness coordinate. As an example, the equations are used to study nonlinear thickness-shear vibrations of a quartz plate driven by an electrical voltage. Nonlinear electrical current amplitude-frequency behavior near resonance is obtained. The equations and results are useful in the study and design of piezoelectric crystal resonators and the measurement of nonlinear material constants of electroelastic materials.**

I. Introduction

For linear piezoelectric plates, there are quite a few versions of two-dimensional theories governing the extensional, flexural, thickness-shear, and thickness-stretch motions for different applications [1]–[4]. These equations have been generalized to include various nonlinear effects for different purposes. Von Karman-type equations for flexure with small strain and large rotation were given in [5]. Equations linear in mechanical variables and nonlinear in electric fields were obtained in [6] for problems involving small deformations and strong electric fields and electrostrictive plates.

Thickness-shear vibrations of piezoelectric plates are widely used as the operating modes of piezoelectric components like quartz resonators and filters [7], gyroscopes (angular rate sensors) [8], and liquid sensors [9]. These devices operate at resonant conditions in which a small, applied electrical voltage can generate a strong mechanical resonance with relatively large thickness-shear deformations. The electric field usually is not strong either because it is generated directly by the small, applied voltage or because it is generated indirectly by mechanical fields through piezoelectric coupling that is weak for materials like quartz. This is different from the case of certain other piezoelectric components like actuators or transformers in which a large voltage is often applied, and a strong electric field is directly generated by the applied voltage. The analysis of piezoelectric devices operating with small electric fields and strong mechanical resonance presents a class of problems of piezoelectric plates with large thickness-shear deformations and weak electric fields. It will be useful to develop a two-dimensional plate theory for this type of problems, which is what the present paper intends to achieve. In Section II a brief summary of the three-dimensional theory of nonlinear electroelasticity is given and approximations relevant to large thickness-shear deformations are stated. Two-dimensional plate equations are derived systematically in Section III. To show the usefulness and effectiveness of the derived equations, the equations are used in an analytical analysis of thickness-shear vibrations of a rotated Y-cut quartz plate in Section IV. Some conclusions are drawn in Section V.

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II. Three-Dimensional Equations of Electroelasticity

Consider an electroelastic body that, in the reference configuration, occupies a region V with boundary surface S. Let N_L be the unit exterior normal of S. The deformation of the body is described by $y_i = y_i(X_L, t)$ where y_i denotes the present coordinates and X_L is the reference coordinates of material points with respect to the same Cartesian coordinate system. The equations of motion and electrostatics are [10], [11]:

$$\begin{aligned} K_{Lk,L} + \rho_0 f_k &= \rho_0 \ddot{y}_k \\ \Delta_{K,K} &= \rho_E, \end{aligned} \quad (1)$$

where ρ_0 is the mass density and ρ_E is the free-charge density per unit undeformed volume, f_k is the body force per unit mass, K_{Lk} is the first Piola-Kirchhoff stress tensor, and Δ_K is the reference electric displacement vector. The Cartesian tensor notation, the summation convention for repeated tensor indices, and the convention that a comma followed by an index denotes partial differentiation with respect to the coordinate associated with the index will be used. A superimposed dot represents material time derivative. Constitutive relations describing material behaviors are given by a free energy density function $\Sigma(E_{KL}, W_K)$ through:

$$K_{Lk} = y_{k,K}\frac{\partial\Sigma}{\partial E_{KL}} + JX_{L,j}\varepsilon_0\left(E_jE_k - \frac{1}{2}E_iE_i\delta_{jk}\right),$$
$$\Delta_K = \varepsilon_0 JX_{K,k}E_k - \frac{\partial\Sigma}{\partial W_K}, \tag{2}$$

where ε_0 is the dielectric permittivity of free space, δ_{jk} is the Kronecker delta, $J = \det(y_{k,K})$, $E_{KL} = (y_{k,K}y_{k,L} - \delta_{KL})/2$ is the finite strain tensor, $E_i = -\phi_{,i}$ the electric field vector, $W_K = -\phi_{,K}$ the reference electric potential gradient, and ϕ is the electric potential. For relatively large mechanical deformations and infinitesimal electric fields, the following energy density is sufficient [12]:

$$\Sigma = \frac{1}{2}c_{ABCD}E_{AB}E_{CD} - e_{ABC}W_AE_{BC} - \frac{1}{2}\chi_{AB}W_AW_B + \frac{1}{6}c_{ABCDEF}E_{AB}E_{CD}E_{EF} + \frac{1}{24}c_{ABCDEFGH}E_{AB}E_{CD}E_{EF}E_{GH}, \tag{3}$$

where c_{ABCD} are the second-order elastic constants, e_{ABC} the piezoelectric constants, and χ_{AB} the dielectric susceptibility that describe linear material behaviors. c_{ABCDEF} and $c_{ABCDEFGH}$ are the third- and fourth-order elastic constants responsible for nonlinear material behavior. The material constants in (3) are called the fundamental material constants. On the boundary surface S of a material body usually the present position y_k or the traction vector $N_LK_{Lj} = t_j$ and the electric potential ϕ or the surface free-charge density $N_K\Delta_K = \sigma$ are prescribed. Denoting $K_{LM} = K_{Lj}\delta_{jM}$, $f_M = f_j\delta_{jM}$, $y_M = y_j\delta_{jM}$, (δ_{jM} is the Kronecker delta in our case when the two Cartesian coordinate systems for X_K and y_k are coincident), introducing the displacement vector $u_K = y_K - X_K$, keeping linear terms of the electric potential gradient and up to cubic terms of the displacement gradient, from (2) and (3) we obtain [12]:

$$K_{LM} = c_{LMRS}u_{R,S} - e_{KLM}W_K + c^e_{LMRSKN}u_{R,S}u_{K,N} + c^e_{LMRSKNIJ}u_{R,S}u_{K,N}u_{I,J},$$
$$\Delta_K = e_{KRS}u_{R,S} + \varepsilon_{KL}W_L, \tag{4}$$

where $\varepsilon_{KL} = \varepsilon_0\delta_{KL} + \chi_{KL}$, and the effective nonlinear elastic constants are defined by:

$$c^e_{LMRSKN} = \frac{1}{2}\left(c_{LMRSKN} + c_{LMNS}\delta_{KR} + c_{LNRS}\delta_{KM}\right),$$
$$c^e_{LMRSKNIJ} = \frac{1}{6}c_{LMRSKNIJ} + \frac{1}{2}\big(c_{LMKNSI}\delta_{RI} + c_{LNSJ}\delta_{MK}\delta_{RI} + c_{LNRSIJ}\delta_{MK}\big). \tag{5}$$

We note that the mechanically nonlinear terms in $(4)_2$ have been dropped for they are multiplied with the small piezoelectric constants [12]. We are interested in thickness-shear vibrations with relatively large shear deformations for a plate whose reference configuration is shown in Fig. 1 with the X_2 axis as the plate normal. The relevant relatively large displacement gradient components are $u_{1,2}$ and $u_{3,2}$. We now introduce a convention that subscripts A, B, and C assume only 1 and 3 but not 2. Keeping nonlinear terms of $u_{1,2}$ and $u_{3,2}$ only [12], from $(4)_1$ we have:

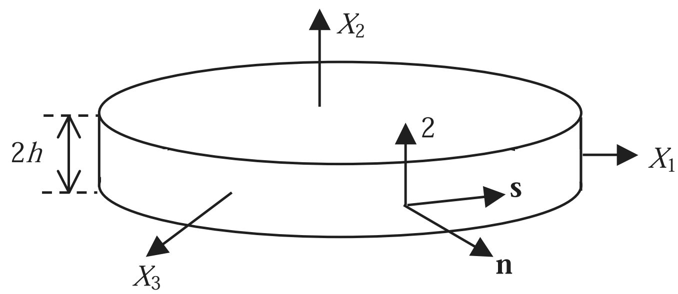

Fig. 1. An electroelastic plate in its reference configuration.

$$K_{LM} = c_{LMRS}u_{R,S} - e_{KLM}W_K + c^e_{LMA2B2}u_{A,2}u_{B,2} + c^e_{LMA2B2C2}u_{A,2}u_{B,2}u_{C,2}. \tag{6}$$

We will use (6) in a variational formulation for nonlinear electroelasticity [13] that is useful in deriving plate equations:

$$\int_V \left[\left(K_{LM,L} + \rho_0 f_M - \rho_0\ddot{y}_M\right)\delta y_M + \Delta_{K,K}\delta\phi\right]\mathrm{d}V + \int_S \left[\left(t_M - N_LK_{LM}\right)\delta y_M + \left(\sigma - N_K\Delta_K\right)\delta\phi\right]\mathrm{d}S = 0, \tag{7}$$

where we have set the body free-charge density ρ_E to zero, which is valid for most applications, and $t_M = t_j\delta_{jM}$.

III. Derivation of Plate Equations

Consider the electroelastic plate in Fig. 1. The plate is assumed to be very thin with its thickness much smaller than its in-plane dimensions.

A. Equations of Motion and Electrostatics

For a first-order plate theory, we make the following expansions of the mechanical displacement and electric potential:

$$u_A \cong u_A^{(0)}(X_1, X_3, t) + X_2u_A^{(1)}(X_1, X_3, t), \quad A = 1, 3,$$
$$u_2 \cong u_2^{(0)}(X_1, X_3, t),$$
$$\phi \cong \phi^{(0)}(X_1, X_3, t) + X_2\phi^{(1)}(X_1, X_3, t), \tag{8}$$

where $u_A^{(0)}$ are the plate extensional displacements, $u_2^{(0)}$ is the flexural displacement, and $u_a^{(1)}$ is the plate thickness-shear displacements. Substituting (8) into (7), with integration by parts, for independent variations of $\delta u_A^{(0)}$,

$\delta u_2^{(0)}$, $\delta u_A^{(1)}$, $\delta\phi^{(0)}$, and $\delta\phi^{(1)}$, we obtain the following two-dimensional equations of motion and electrostatics in a way similar to [1]–[4]:

$$\begin{aligned}
&K_{BM,B}^{(0)} + F_M^{(0)} = 2\rho_0 h \ddot{u}_M^{(0)}, \quad M = 1,2,3,\\
&K_{BA,B}^{(1)} - K_{2A}^{(0)} + F_A^{(1)} = \frac{2\rho_0 h^3}{3}\ddot{u}_A^{(1)}, \quad A = 1,3,\\
&\Delta_{A,A}^{(0)} + \sigma^{(0)} = 0,\\
&\Delta_{A,A}^{(1)} - \Delta_2^{(0)} + \sigma^{(1)} = 0.
\end{aligned} \tag{9}$$

In $(9)_1$ for $M = 1$ and 3 are the equations for extension, and for $M = 2$ the equation for flexure. In $(9)_2$ are for thickness-shear in the X_1 and X_3 directions. In (9) the plate resultants and surface as well as body loads of various orders are defined by:

$$\begin{aligned}
&\left\{K_{LM}^{(n)}, \Delta_K^{(n)}\right\} = \int_{-h}^{h} X_2^n \left\{K_{LM}, \Delta_K\right\} dX_2,\\
&F_M^{(n)} = \left[X_2^n K_{2M}\right]_{-h}^{h} + \int_{-h}^{h} X_2^M \rho_0 f_M dX_2,\\
&\sigma^{(n)} = \left[X_2^n \Delta_2\right]_{-h}^{h}, \quad n = 0,1,
\end{aligned} \tag{10}$$

where $K_{LM}^{(n)}$ represent plate extensional and shearing forces, and bending and twisting moments.

B. Displacement and Potential Gradients

From (8) we can write:

$$\begin{aligned}
u_{R,S} &= U_{RS}^{(0)} + X_2 U_{RS}^{(1)},\\
W_K &= -\phi_{,K} = W_K^{(0)} + X_2 W_K^{(1)},
\end{aligned} \tag{11}$$

where

$$\begin{aligned}
&U_{11}^{(0)} = u_{1,1}^{(0)}, \ U_{12}^{(0)} = u_1^{(1)}, \ U_{13}^{(0)} = u_{1,3}^{(0)},\\
&U_{21}^{(0)} = u_{2,1}^{(0)}, \ U_{22}^{(0)} = 0, \ U_{23}^{(0)} = u_{2,3}^{(0)},\\
&U_{31}^{(0)} = u_{3,1}^{(0)}, \ U_{32}^{(0)} = u_3^{(1)}, \ U_{33}^{(0)} = u_{3,3}^{(0)},\\
&U_{11}^{(1)} = u_{1,1}^{(1)}, \ U_{12}^{(1)} = 0, \ U_{13}^{(1)} = u_{1,3}^{(1)},\\
&U_{21}^{(1)} = 0, \ U_{22}^{(1)} = 0, \ U_{23}^{(1)} = 0,\\
&U_{31}^{(1)} = u_{3,1}^{(1)}, \ U_{32}^{(1)} = 0, \ U_{33}^{(1)} = u_{3,3}^{(1)},
\end{aligned} \tag{12}$$

and

$$\begin{aligned}
&W_1^{(0)} = -\phi_{,1}^{(0)}, \ W_2^{(0)} = -\phi^{(1)}, \ W_3^{(0)} = -\phi_{,3}^{(0)},\\
&W_1^{(1)} = -\phi_{,1}^{(1)}, \ W_2^{(1)} = 0, \ W_3^{(1)} = -\phi_{,3}^{(1)}.
\end{aligned} \tag{13}$$

C. Constitutive Relations

Because the plate is assumed to be thin, we make the stress relaxation assumption of vanishing normal stress $K_{22} = 0$. This implies, through (6) by setting $L = M = 2$, the following expression for $u_{2,2}$ in terms of other components of the displacement and potential gradients:

$$\begin{aligned}
u_{2,2} = -\frac{1}{c_{2222}}\big(&c_{22RS}u_{R,S} - c_{2222}u_{2,2} - e_{K22}W_K\\
&+ c_{22A2B2}^e u_{A,2}u_{B,2} + c_{22A2B2C2}^e u_{A,2}u_{C,2}\big).
\end{aligned} \tag{14}$$

We note that stress relaxations for thin anisotropic or piezoelectric plates can be made in ways more sophisticated than the above, also involving K_{21} and K_{32} [1]–[4]. That is not our main interest here. The above relaxation involving K_{22} is the major relaxation because, in anisotropic plates, couplings among extensions in different directions usually are much larger than couplings between extensions and shears or other couplings. We also note that in (14) $u_{2,2}$ has been eliminated from the right-hand side of (14) because, when $R = S = 2$, the two terms containing $u_{2,2}$ will cancel each other. From (14) the thickness expansion or contraction accompanying the extension and flexure of the plate due to Poisson's effect can be found if wanted. Substituting (14) back into (6) and $(4)_2$, we obtain the following constitutive relations relaxed for thin plates:

$$\begin{aligned}
K_{LM} &= \bar{c}_{LMRS}u_{R,S} - \bar{e}_{KLM}W_K + \bar{c}_{LMA2B2}u_{A,2}u_{B,2}\\
&\quad + \bar{c}_{LMA2B2C2}u_{A,2}u_{B,2}u_{C,2},\\
\Delta_K &= \bar{e}_{KRS}u_{R,S} + \bar{\varepsilon}_{KL}W_L,
\end{aligned} \tag{15}$$

where the plate material constants are defined by:

$$\begin{aligned}
&\bar{c}_{LMRS} = c_{LMRS} - c_{LM22}c_{22RS}/c_{2222},\\
&\bar{c}_{LMA2B2} = c_{LMA2B2}^e - c_{LM22}c_{22A2B2}^e/c_{2222},\\
&\bar{c}_{LMA2B2C2} = c_{LMA2B2C2}^e - c_{LM22}c_{22A2B2C2}^e/c_{2222},\\
&\bar{e}_{KLM} = e_{KLM} - c_{LM22}e_{K22}/c_{2222},\\
&\bar{\varepsilon}_{KL} = \varepsilon_{KL} + e_{K22}e_{L22}/c_{2222}.
\end{aligned} \tag{16}$$

We note that the right-hand side of (15) does not contain $u_{2,2}$ and $K_{22} = 0$ is automatically satisfied by (15). Integrating (15) through the plate thickness, we obtain the zero-order, two-dimensional plate constitutive relations below in $(17)_{1,2}$. Multiplying both sides of (15) by X_2 and integrating the resulting equation through the plate thickness we have the first-order plate constitutive relations in $(17)_{3,4}$:

$$\begin{aligned}
K_{LM}^{(0)} &= 2h\Big(c_{LMRS}^{(0)}U_{RS}^{(0)} - e_{KLM}^{(0)}W_K^{(0)} + \bar{c}_{LMA2B2}u_A^{(1)}u_B^{(1)}\\
&\qquad + \bar{c}_{LMA2B2C2}u_A^{(1)}u_B^{(1)}u_c^{(1)}\Big),\\
\Delta_K^{(0)} &= 2h\left(e_{KRS}^{(0)}U_{RS}^{(0)} + \bar{\varepsilon}_{KL}W_L^{(0)}\right),\\
K_{LM}^{(1)} &= \frac{2h^3}{3}\left(\bar{c}_{LMRS}U_{RS}^{(1)} - \bar{e}_{KLM}W_K^{(1)}\right),\\
\Delta_K^{(1)} &= \frac{2h^3}{3}\left(\bar{e}_{KRS}U_{RS}^{(1)} + \bar{\varepsilon}_{KL}W_L^{(1)}\right),
\end{aligned} \tag{17}$$

where, following Mindlin [2], we have modified the zero-order linear plate constants by the introduction of two shear correction factors κ_1 and κ_3 as follows [2]:

$$\begin{aligned} c^{(0)}_{IJKL} &= \kappa^{\mu}_{I+J-2}\kappa^{v}_{K+L-2}\bar{c}_{IJKL}, \\ e^{(0)}_{KIJ} &= \kappa^{\mu}_{I+J-2}\bar{e}_{KIJ}, \ \text{(not summed)}, \\ \mu &= \cos^2(IJ\pi/2), \ \ v = \cos^2(KL\pi/2). \end{aligned} \tag{18}$$

Thus κ^{μ}_{I+J-2} (or κ^{v}_{K+L-2}) is equal to κ_1, κ_3, or unity according as $I+J$ (or $K+L$) is 3, 5, or neither, respectively. These two correction factors should be determined by requiring the two fundamental thickness-shear resonant frequencies obtained from the linear version of the above two-dimensional plate equations to be equal to the corresponding exact frequencies predicted by the three-dimensional linear equations. With shear correction factors thus determined, the two-dimensional plate equations above and the exact three-dimensional equations yield the same frequencies for a particular motion, i.e., the linear thickness-shear vibrations of a plate in the two fundamental thickness-shear modes.

D. Boundary Conditions

We have obtained the two-dimensional equations of motion and electrostatics (9), the constitutive relations (17) and (18), and the displacement and potential gradients (12) and (13). With successive substitutions, (9) can be written as seven equations for the seven unknowns of $u^{(0)}_1$, $u^{(0)}_2$, $u^{(0)}_3$, $u^{(1)}_1$, $u^{(1)}_3$, $\phi^{(0)}$, and $\phi^{(1)}$. To these equations the proper forms of the boundary conditions can be determined from the variational formulation (7) in a way similar to [1]–4]. At the boundary of a plate with unit exterior normal $\mathbf{N}$ and unit in-plane tangent $\mathbf{S}$ in the reference configuration, we need to prescribe:

$$\begin{aligned} &K^{(0)}_{NN} \ \text{or} \ u^{(0)}_{N}, \ \ K^{(0)}_{NS} \ \text{or} \ u^{(0)}_{S}, \\ &K^{(0)}_{N2} \ \text{or} \ u^{(0)}_{2}, \\ &K^{(1)}_{NN} \ \text{or} \ u^{(1)}_{N}, \ \ K^{(1)}_{NS} \ \text{or} \ u^{(1)}_{S}, \\ &\Delta^{(0)}_{N} \ \text{or} \ \phi^{(0)}, \ \text{or} \ \Delta^{(1)}_{N} \ \text{or} \ \phi^{(1)}. \end{aligned} \tag{19}$$

With accurate finite element results from the three-dimensional equations, it is now known that the first-order Mindlin's plate equations with correction factors does not yield accurate enough results for plates with certain length-to-thickness ratios [14], although the infinite plate thickness-shear frequencies can be made exact by the correction factors. This inaccuracy may be of concern when the frequency precision requirement is high. However, because a plate theory is much simpler than the three-dimensional equations, analytical solutions often can be obtained from plate equations. Even when the accuracy in frequency prediction is not high, simple analytical solutions showing the mechanisms of physical phenomena, and qualitative behaviors of plate vibrations are still very valuable and useful. The above nonlinear plate equations may allow simple analytical solutions to many phenomena that require complicated or even impossible analytical analyses from the three-dimensional equations. In Section IV we study the nonlinear, thickness-shear vibration of a quartz plate as an example to show the simplicity when plate equations are used.

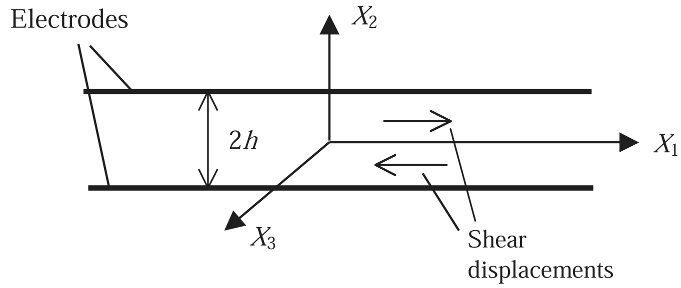

Fig. 2. An electroded rotated Y-cut quartz plate.

IV. Large Thickness-Shear Vibration of a Rotated Y-cut Quartz Plate

As an example for the applications of the two-dimensional equations derived above, we now study large thickness-shear vibration in the X_1 direction of a rotated Y-cut quartz plate, which is a widely used operating mode of piezoelectric resonators [7], [15]. Rotated Y-cut quartz plates effectively have the symmetry of monoclinic crystals for which, under the compact matrix notation [7], the second-order materials constants for linear material behaviors can be represented by the following matrices:

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & 0 & 0 \\ c_{21} & c_{22} & c_{23} & c_{24} & 0 & 0 \\ c_{31} & c_{32} & c_{33} & c_{34} & 0 & 0 \\ c_{41} & c_{42} & c_{43} & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & c_{56} \\ 0 & 0 & 0 & 0 & c_{65} & c_{66} \end{pmatrix}, \begin{pmatrix} e_{11} & 0 & 0 \\ e_{12} & 0 & 0 \\ e_{13} & 0 & 0 \\ e_{14} & 0 & 0 \\ 0 & e_{25} & e_{35} \\ 0 & e_{26} & e_{36} \end{pmatrix} \begin{pmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{22} & \varepsilon_{23} \\ 0 & \varepsilon_{32} & \varepsilon_{33} \end{pmatrix}. \tag{20}$$

We consider a plate electroded at its two major faces (Fig. 2). The electrodes are assumed to be very thin so their mechanical effects can be neglected. For thickness-shear in the X_1 direction $u^{(1)}_1$ is the dominating mechanical displacement that is coupled to $\phi^{(1)}$ due to (20). We consider a very thin plate so that edge effects can be neglected, and the thickness-shear mode we are considering does not vary with X_1 and X_3. Then the relevant equations from Section III are:

$$\begin{aligned} -K^{(0)}_{21} &= \frac{2\rho_0 h^3}{3}\ddot{u}^{(1)}_1, \\ K^{(0)}_{21} &= 2h\Big(c^{(0)}_{2112}U^{(0)}_{12} - e^{(0)}_{221}W^{(0)}_2 + \bar{c}_{211212}u^{(1)}_1 u^{(1)}_1 \\ &\qquad + \bar{c}_{21121212}u^{(1)}_1 u^{(1)}_1 u^{(1)}_1\Big), \\ \Delta^{(0)}_2 &= 2h\left(e^{(0)}_{212}U^{(0)}_{12} + \bar{\varepsilon}_{22}W^{(0)}_2\right), \\ U^{(0)}_{12} &= u^{(1)}_1, \\ W^{(0)}_1 &= -\phi^{(1)}. \end{aligned} \tag{21}$$

From $(21)_{2,4,5}$ the equation of motion $(21)_1$ can be written as:

$$\ddot{u} + \omega_0^2 u + \beta u^2 + \gamma u^3 = \alpha\phi^{(1)}. \tag{22}$$

In (22) we have denoted:

$$\begin{aligned} & u = u_1^{(1)},\ \omega_0^2 = 3c_{2112}^{(0)}/\left(\rho_0 h^2\right), \\ & \alpha = -3e_{221}^{(0)}/\left(\rho_0 h^2\right), \\ & c_{2112}^{(0)} = \kappa_1^2 \overline{c}_{2112},\ e_{221}^{(0)} = \kappa_1 \overline{e}_{221}, \\ & \kappa_1^2 = \frac{\pi^2}{12}\left(1 - \frac{8k_{26}^2}{\pi^2}\right), \\ & k_{26}^2 = \frac{e_{26}^2}{\varepsilon_{22}\hat{c}_{66}},\ \hat{c}_{66} = c_{66} + \frac{e_{26}^2}{\varepsilon_{22}}, \\ & \beta = 3\overline{c}_{211212}/\left(\rho_0 h^2\right),\ \gamma = 3\overline{c}_{21121212}/\left(\rho_0 h^2\right), \end{aligned} \tag{23}$$

where the thickness-shear correction factor is taken from [2]. In (23), ω_0 is the fundamental thickness-shear frequency from a linear solution. We want to study free and forced vibrations near ω_0. We note that the fundamental thickness-shear frequency in (23), as predicted by the linear portion of the plate equation (22), is exactly the same as the prediction from the exact three-dimensional solution due to the introduction of correction factors.

A. Free Vibration

We look for a periodic solution with undetermined frequency ω and undetermined amplitude A to the homogeneous form of (22). For monoclinic crystals, $\beta = 0$, and therefore the quadratic nonlinear term in (22), disappears. Substituting $u = A\cos\omega t$ into (22) (with $V = 0$), neglecting the $\cos 3\omega t$ term [16], we obtain:

$$-\omega^2 A + \omega_0^2 A + \frac{3}{4}\gamma A^3 = 0. \tag{24}$$

The (24) yields the following expression for the nonlinear resonant frequency:

$$\omega = \sqrt{\omega_0^2 + \frac{3}{4}\gamma A^2} \cong \omega_0\left(1 + \frac{3\gamma A^2}{8\omega_0^2}\right), \tag{25}$$

and the following corresponding free vibration mode:

$$u = A\cos\omega_0\left(1 + \frac{3\gamma A^2}{8\omega_0^2}\right)t. \tag{26}$$

We note that (26) as obtained from the above procedure used in [16] is the same as the result obtained from the various asymptotic procedures in [17]. In (26) is shown the familiar and important behavior that for large amplitude vibration the frequency becomes amplitude dependent.

B. Forced Vibration

We consider electrically forced vibrations driven by a voltage across the thickness of the plate with $\phi(\pm h) = \pm 0.5V\cos\omega t$. Then $(8)_3$ implies:

$$\phi^{(0)} = 0,\ \phi^{(1)} = \frac{\alpha V}{2h}\cos\omega t. \tag{27}$$

With $\phi^{(1)}$ from (27), (22) can be written as:

$$\ddot{u} + \omega_0^2 u + c\dot{u} + \gamma u^3 = \frac{\alpha V}{2h}\cos\omega t, \tag{28}$$

where we also have introduced a damping term with damping coefficient $c = 2\omega_0\zeta = \omega_0/Q$, where ζ is the relative damping coefficient and Q is the quality factor. We look for a solution to (28) in the following form [16]:

$$u = A\cos(\omega t + \psi), \tag{29}$$

where A and ψ are undetermined constants. Substituting (29) into (28), neglecting the $\cos 3(\omega t + \psi)$ term [16], collecting coefficients of $\sin\omega t$ and $\cos\omega t$, we obtain:

$$\begin{cases} \left[\left(\omega_0^2 - \omega^2\right)A + \frac{3}{4}\gamma A^3\right]\cos\psi - cA\omega\sin\psi = \frac{\alpha V}{2h}, \\ \left[\left(\omega_0^2 - \omega^2\right)A + \frac{3}{4}\gamma A^3\right]\sin\psi + cA\omega\cos\psi = 0. \end{cases} \tag{30}$$

Multiplying $(30)_{1,2}$ by $\cos\psi$ and $\sin\psi$ then adding them and multiplying $(30)_{1,2}$ by $\sin\psi$ and $\cos\psi$ and subtracting one from the other, we have:

$$\begin{cases} \left(\omega_0^2 - \omega^2\right)A + \frac{3}{4}\gamma A^3 = \frac{\alpha V}{2h}\cos\psi, \\ cA\omega = -\frac{\alpha V}{2h}\sin\psi. \end{cases} \tag{31}$$

Squaring both sides of $(31)_{1,2}$ and adding them yields:

$$\left[\left(\omega_0^2 - \omega^2\right)A + \frac{3}{4}\gamma A^3\right]^2 + (cA\omega)^2 = \left(\frac{\alpha V}{2h}\right)^2. \tag{32}$$

We are interested in resonant behaviors near ω_0. Therefore, we denote $\omega = \omega_0 + \Delta\omega$. Then $\omega_0^2 - \omega^2 \cong -2\omega_0\Delta\omega$ and (32) can be written as:

$$\left(\Delta\omega - \frac{3\gamma A^2}{8\omega_0}\right)^2 + \frac{c^2}{4} = \left(\frac{\alpha V}{4h\omega_0 A}\right)^2, \tag{33}$$

from which we can solve for $\Delta\omega$:

$$\Delta\omega = \frac{3\gamma A^2}{8\omega_0} \pm \frac{1}{2}\left[\left(\frac{\alpha V}{2h\omega_0 A}\right)^2 - c^2\right]^{\frac{1}{2}}. \tag{34}$$

We are interested in the electric current flowing in or out of the driving electrodes, which is important to resonator design. From $(21)_3$, $(23)_1$, and (29), we have for the free electric charge per unit undeformed area of the electrode at $X_2 = h$:

$$\begin{aligned} Q_e = -\frac{\Delta_2^{(0)}}{2h} &= -\left(\kappa_1\overline{e}_{26}u - \overline{\varepsilon}_{22}\phi^{(1)}\right) \\ &\cong -\kappa_1\overline{e}_{26}u = -\kappa_1\overline{e}_{26}A\cos(\omega t + \psi), \end{aligned} \tag{35}$$

where, for near resonance behavior, we have neglected the electrostatic term in the expression of Q_e, which is much smaller than the piezoelectric term [16]. Then the current flows out of the electrode are:

$$
\begin{aligned}
&i = -\dot{Q}_e = -\kappa_1 \bar{e}_{26} A\omega \sin(\omega t + \psi) = I \sin(\omega t + \psi), \\
&I = -\kappa_1 \bar{e}_{26} A\omega \cong -\kappa_1 \bar{e}_{26} A\omega_0.
\end{aligned} \tag{36}
$$

From $(36)_2$ and (34), we obtain the frequency-current amplitude relation of interest:

$$
\Delta\omega = \frac{3\gamma}{8\omega_0}\left(\frac{I}{\kappa_1 \bar{e}_{26}\omega_0}\right)^2 \pm \frac{1}{2}\left[\left(\frac{\alpha V \kappa_1 \bar{e}_{26}}{2hI}\right)^2 - c^2\right]^{1/2}. \tag{37}
$$

Expressions similar to (37) were obtained in [16], [18] from the three-dimensional equations of nonlinear electroelasticity. The above procedure using two-dimensional equations is much simpler.

Next we consider a numerical example of an AT-cut quartz plate that is a rotated Y-cut quartz plate with $\theta = 35.25°$. For quartz $\rho = 2649$ kg/m^3 and for an AT-cut quartz plate [7]:

$$
\begin{aligned}
&\begin{pmatrix}
86.75 & -8.25 & 27.15 & -3.66 & 0 & 0 \\
-8.25 & 129.77 & -7.42 & 5.7 & 0 & 0 \\
27.15 & -7.42 & 102.83 & 9.92 & 0 & 0 \\
-3.66 & 5.7 & 9.92 & 38.61 & 0 & 0 \\
0 & 0 & 0 & 0 & 68.81 & 2.53 \\
0 & 0 & 0 & 0 & 2.53 & 29.01
\end{pmatrix} \times 10^9 \left(N/m^2\right), \\
&\begin{pmatrix}
0.171 & 0 & 0 \\
-0.152 & 0 & 0 \\
-0.0187 & 0 & 0 \\
0.067 & 0 & 0 \\
0 & 0.108 & -0.0761 \\
0 & -0.095 & 0.067
\end{pmatrix} \left(C/m^2\right), \\
&\begin{pmatrix}
39.21 & 0 & 0 \\
0 & 39.82 & 0.86 \\
0 & 0.86 & 40.42
\end{pmatrix} \times 10^{-12} (F/m).
\end{aligned} \tag{38}
$$

From (5), (16), and the Appendix of [19] for monoclinic crystals in addition to $\beta = 0$ we also have:

$$
c^e_{6666} = \frac{1}{6} c_{6666} + \frac{1}{2}\left(c_{22} + 2c_{266}\right), \ \ \bar{c}_{6666} = c^e_{6666}, \tag{39}
$$

where the compact matrix indices are used. From [12] we have:

$$
\begin{aligned}
&\frac{1}{6} c_{6666} + \frac{1}{2}\left(c_{22} + 2c_{266}\right) = 13.7 \times 10^{11} \ N/m^2, \\
&c_{6666} = 77 \times 10^{11} \ N/m^2.
\end{aligned} \tag{40}
$$

Quartz is a material with very little damping. We choose the quality factor to be $Q = 10^5$. We consider a 1 MHz fundamental mode resonator with $h = 0.8273$ mm. The frequency-current amplitude relation predicted by (37) is plotted in Fig. 3, which is typical for this type of nonlinear resonance [16].

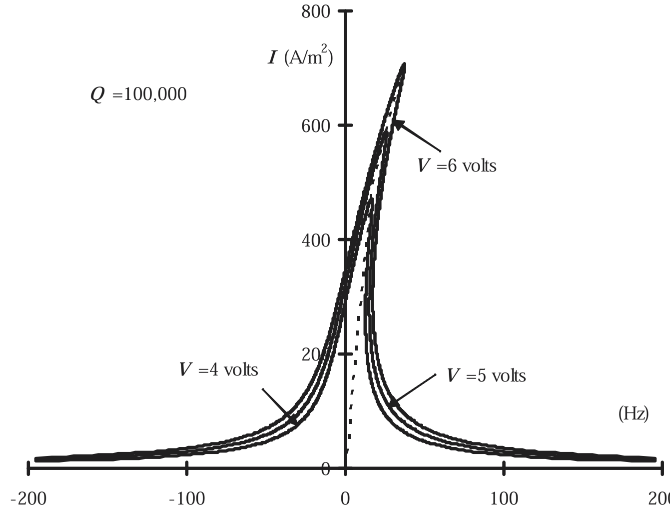

Fig. 3. Nonlinear amplitude-frequency behavior near resonance.

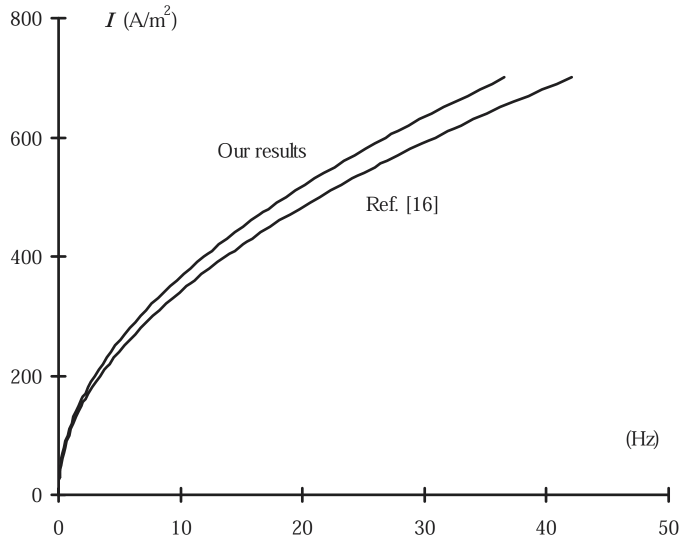

Fig. 4. Amplitude-frequency behavior predicted by two- and three-dimensional equations.

The vertices of the curves in Fig. 3 fall on a parabola as predicted by the three-dimensional equations [16]. Equations for the corresponding parabola predicted by the two-dimensional equations in the present paper is (37) without the last term on the right-hand side. In Fig. 4 we plot the two parabolas from the two- and three-dimensional solutions as a comparison. It can be seen that the two parabolas are very close to each other.

V. Conclusions

Two-dimensional nonlinear equations for electroelastic plates in relatively large thickness-shear deformations are obtained. Quadratic and cubic effects of the large thickness-shear deformations are included. The third- and fourth-order elastic constants are involved. The equations represent a much simpler mathematical model than the three-dimensional equations. An example shows that simple, analytical analysis is possible using these equations, and the results are close to what the three-dimensional theory predicts. The equations are useful in the design of piezoelectric resonators, filters, and other devices operating with thickness-shear modes. The equations also can be discretized for numerical analysis of complicated problems, as what had been done extensively for linear plate equations.

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